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Hierarchical low-rank approximation of tensors by successive rank one corrections for preconditioning and solving high dimensional linear systems

L. Giraldi*, A. Nouy, G. Legrain

LUNAM Université, GeM, UMR CNRS 6183, École Centrale de Nantes, Université de Nantes
1 rue de la Noë 44300 Nantes, France
*Loic.Giraldi@ec-nantes.fr

Abstract. *We propose an algorithm for preconditioning and solving high dimensional linear systems of equations in tensor format. The algorithm computes an approximation of a tensor in hierarchical Tucker format in a subspace constructed from successive rank one corrections. The algorithm can be applied for the approximation of a solution or of the inverse of an operator. In the latter case, properties such as sparsity or symmetry can be imposed to the approximation. The methodology is applied to high dimensional problems arising from the discretization of stochastic parametric problems.*

Keywords: high-dimensional problems; low-rank tensor approximation; hierarchical Tucker format; proper generalized decomposition; preconditioning.

Introduction

We are interested in linear systems in high order tensor spaces, such as the ones arising from the discretization of stochastic parametric partial differential equations. Such systems are of the form

$$Au = b, \tag{1}$$

where u and b belong to a finite dimensional tensor space $\mathcal{V} = \bigotimes_{\mu=1}^{d} \mathcal{V}^{\mu} \simeq \bigotimes_{\mu=1}^{d} \mathbb{R}^{n_\mu}$ and $A \in \mathcal{L}(\mathcal{V}) = \bigotimes_{\mu=1}^{d} \mathcal{L}(\mathcal{V}^{\mu}) \simeq \bigotimes_{\mu=1}^{d} \mathbb{R}^{n_\mu \times n_\mu}$. In order to circumvent the curse of dimensionality, methods based on low-rank approximations have recently been proposed. The first class of methods consists in introducing low-rank tensor approximation methods in classical iterative solvers [1, 2]. This approach is simple but generally requires good preconditioners in low-rank tensor format. We can find in the literature rank one preconditioners [3, 4, 5, 6] and predefined rank r preconditioners [7, 8, 9]. The second approach, sometimes called Proper Generalized Decomposition (PGD), tries to find a direct approximation of the solution in low-rank tensor sets [10, 11, 12].

Here, we propose a new algorithm for the progressive construction of low-rank approximations in hierarchical Tucker tensor format. This algorithm can be applied for the direct approximation of the solution of a linear system and also for obtaining an approximation of the inverse of an operator that can be used as a preconditioner.

Tensor spaces and low-rank subsets

Let $d \geq 2$ and $D = \{1, \dots, d\}$. We consider the finite dimensional Hilbert space $\mathcal{X}^{\mu}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mu}$ and the associated norm $\|\cdot\|_{\mu}$, for $\mu \in D$. The space $\mathcal{X} = \bigotimes_{\mu} \mathcal{X}^{\mu}$, where $\bigotimes_{\mu}$ stands for $\bigotimes_{\mu=1}^{d}$, is a Hilbert space equipped with the induced inner product $\langle \cdot, \cdot \rangle$ defined for rank one tensors by $\left\langle \bigotimes_{\mu} x^{\mu}, \bigotimes_{\mu} y^{\mu} \right\rangle = \prod_{\mu \in D} \langle x^{\mu}, y^{\mu} \rangle_{\mu}$ and extended to $\mathcal{X}$ by linearity. The norm associated to $\langle \cdot, \cdot \rangle$ is noted $\|\cdot\|$. We consider the set of rank r tensors, denoted $\mathcal{C}_r(\mathcal{X})$ and defined by

$$\mathcal{C}_r(\mathcal{X}) = \left\{ x = \sum_{i=1}^{r} \bigotimes_{\mu} x_i^{\mu};\ x_i^{\mu} \in \mathcal{X}^{\mu} \right\}.$$

The set of rank-r Tucker tensors, with $r = (r_1, \dots, r_d)$, contains tensors of the form

$$x = \sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_d} \alpha_{i_1 \dots i_d} \bigotimes_{\mu} x^{\mu}_{i_\mu}. \tag{2}$$

This format has nice approximation properties but with a core tensor $\alpha \in \bigotimes_\mu \mathbb{R}^{r_\mu}$, it again suffers from the curse of dimensionality. Letting T be a dimension tree on D and $r = (r_t)_{t \in T}$ be a set of integers, the set $\mathcal{H}^T_r(\mathcal{X})$ of rank-r Hierarchical Tucker tensors contains tensors of the form (2) where the tensor $\alpha \in \mathcal{H}^T_r(\bigotimes_\mu \mathbb{R}^{r_\mu})$ has a low-rank tensor structure. More precisely, for $t \in T$, the t-matricization $\mathscr{M}^t(\alpha)$ of α has a rank less than r_t, where $\mathscr{M}^t(\alpha)_{(i_\mu, \mu \in t)(i_\mu, \mu \in D \setminus t)} = \alpha_{i_1, \dots, i_d}$.

Algorithm for the approximation of a tensor in hierarchical format

We are interested in approximating a tensor $x \in \mathcal{X}$ with respect to a certain norm $\|\cdot\|_{\mathcal{X}}$ by solving the problem

$$\inf_{y \in \mathcal{M}} \|x - y\|^2_{\mathcal{X}}$$

When the norm is not an induced norm, this minimization problem can not be solved using standard SVD based algorithms [13].
The proposed algorithm consists in constructing a sequence of approximations $x^{(k)}$ in a sequence of approximation spaces $\mathcal{U}^{(k)} = \bigotimes_\mu \mathcal{U}^{\mu,(k)}$, where the $\{\mathcal{U}^{\mu,(k)}\}_{k \geq 0}$ form an increasing sequence of k-dimensional spaces in $\mathcal{X}^\mu$ which is constructed from successive rank-one corrections of the iterates $x^{(k)}$. More precisely, the algorithm is as follows: starting from $x^{(0)}$, for all $k \geq 1$, do

1. Compute $z^{(k)} = \bigotimes_\mu z^{\mu,(k)}$ by solving $\min_{y \in \mathcal{C}_1(\mathcal{X})} \left\|x - x^{(k-1)} - y\right\|_{\mathcal{X}}$
2. Set $\mathcal{U}^{\mu,(k)} = \mathcal{U}^{\mu,(k-1)} + \operatorname{span}\left\{z^{\mu,(k)}\right\}$
3. Compute $x^{(k)}$ by solving $\min_{y \in \mathcal{H}^T_{r^{(k)}}(\mathcal{U}^{(k)})} \|x - y\|_{\mathcal{X}}$

Steps 1 and 3 are solved via an alternating minimization algorithm. We can show that the sequence $\left(x^{(k)}\right)_{k \geq 1}$ converges toward x if the problem in step 1 is solved exactly [14].

If we set $\mathcal{X} = \mathcal{V}$, and $\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{A^*A}$ (or $\|\cdot\|_A$ if A is symmetric positive definite) a norm induced by the operator, the algorithm enables to compute an approximation of the solution of the linear system (1).
Now, for $\mathcal{X} = \mathcal{L}(\mathcal{V})$, if $\|\cdot\|_{\mathcal{X}}$ is the norm induced by the inner product $\langle \cdot AA^*, \cdot \rangle$ (resp. $\langle \cdot A, \cdot \rangle$), one can compute a left approximate inverse of the definite operator A (resp. symmetric definite positive operator A). In order to impose properties to the approximation, we introduce in step 1 a minimization in the subset $\mathcal{C}_1(\widetilde{\mathcal{X}}) \subset \mathcal{C}_1(\mathcal{X})$ where $\widetilde{\mathcal{X}}$ is a space of operators satisfying particular properties such that symmetries or sparsities. For imposing symmetry, we need to solve a Sylvester equation at each step of the alternate algorithm. For imposing sparsity, we use an adaptation of the SParse Approximate Inverse (SPAI) method which allows the adaptive construction of patterns [15].

Application

The methodology will be illustrated on high dimensional problems arising from the discretization of stochastic equations.

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